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Phase Transition for Absorbed Brownian Motion with Drift

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We study one-dimensional Brownian motion with constant drift toward the origin and initial distribution concentrated in the strictly positive real line. We say that at the first time the process hits the origin, it is absorbed. We study the asymptotic behavior, as $t \to \infty$, of m_t, the conditional distribution at time zero of the process conditioned on survival up to time t and on the process having a fixed value at time t. We find that there is a phase transition in the decay rate of the initial condition. For fast decay rate (subcritical case) m_t is localized, in the critical case m_t is located around $\sqrt{t}$, and for slow rates (supercritical case) m_t is located around t. The critical rate is given by the decay of the minimal quasistationary distribution of this process. We also study in each case the asymptotic distribution of the process, scaled by $\sqrt{t}$, conditioned as before. We prove that in the subcritical case this distribution is a Brownian excursion. In the critical case it is a Brownian bridge attaining 0 for the first time at time 1, with some initial distribution. In the supercritical case, after centering around the expected value—which is of the order of t—we show that this process converges to a Brownian bridge arriving at 0 at time 1 and with a Gaussian initial distribution.

KEY WORDS: Absorbed Brownian motion; quasistationary distributions; conditioned Brownian motion with drift.

1. INTRODUCTION

Let $(B_t)_{t \geqslant 0}$ be a standard Brownian motion defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$. Take $\alpha > 0$ and consider

$$X_t = B_t - \alpha t$$

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a Brownian motion with constant drift $-\alpha$. By $\mathbb{P}_x$ we mean the distribution law of the process with initial condition $X_0 = x$.

Let ν be a measure concentrated on $(0, \infty)$. We denote by $\mathbb{P}_\nu$ the distribution of the process when the initial point X_0 is chosen with the measure ν. Let T_0^X be the hitting time of the origin, that is, $T = \inf\{t: X_t = 0\}$. Informally we say that at this time the process is absorbed in 0.

One of the problems studied in absorbing Markov processes is the behavior of the process conditioned on nonabsorption. That is, one studies the law

$$\mathbb{P}_\nu\{\bullet \mid T_0^X > t\}$$

To our knowledge there are no attempts in the literature to study the conditioned process in the whole interval $[0, t]$. We want to take advantage of the fact that one can compute almost everything in the Brownian motion setting to show the kind of results that one can expect in more general cases.

We consider initial distributions ν on $(0, \infty)$ satisfying three conditions (the classification will be clear in a moment):

- Subcritical case: $\int xe^{\alpha x}\, \nu(dx) < \infty$.
- Critical case: $d\nu/dx = kx^m e^{-\alpha x}$ for some $m \geq 0$.
- Supercritical case: $d\nu/dx = h(x)\, e^{-\theta x}$, where $\theta \in (0, \alpha)$ and h is, for instance, a polynomial in x.

In Theorems 1 and 2 we study the law of the process $(X_s)_{s \in [0, t]}$ conditioned on $T_0^X > t$. In the subcritical and critical cases we rescale space dividing by $\sqrt{t}$. In the subcritical case we show that the law of the process $(X_{st}/\sqrt{t})_{s \in [0, 1]}$ conditioned on $T_0^X > t$ converges to the law of a Brownian excursion. That is, it converges to a Brownian motion conditioned to stay positive in $(0, 1)$ and to be at the origin at times 0 and 1. In the critical case we show that the law of the above rescaled and conditioned process converges to the law of a Brownian bridge with initial distribution (proportional to) $x^{m+1}e^{-x^2/2}$ conditioned to stay positive in $[0, 1)$ and to be at the origin at time 1.

In the supercritical case we change the normalization and show that $(X_{st}/t)_{s \in [0, 1]}$ conditioned on $T_0^X > t$ converges in distribution to the deterministic motion that follows the line $y(s) = (\alpha - \theta)(1 - s)$. To see the fluctuations around this line we study the process

$$\left(\frac{X_{st} - y(s)}{\sqrt{t}}\right)_{s \in [0, 1]}$$

and show that it converges to Brownian motion conditioned to be at 0 at time 1 with a Gaussian initial distribution.

We conjecture that the same kind of results can be shown for the asymmetric random walk on $\mathbb{N}$. This process makes jumps of length one with probability p to the right and q to the left, with $q>p$. The minimal qsd and the Yaglom limit were obtained by Seneta.[8]

In contrast, for a family of probabilistic cellular automata that contains subcritical oriented percolation, Ferrari *et al.*[4] obtained that the process starting at a fixed configuration conditioned to nonabsorption until t stays essentially in a finite set of states for all times $s \leqslant t$. Hence, if we call X_t the number of infected points at time t, then $(X_{st}/\sqrt{t})_{s\in[0,1]}$ conditioned on $T_0^X>t$ converges to the deterministic trajectory $y(s)\equiv 0$. The same must be true for the subcritical branching process. It would be interesting to understand how these models behave in the critical and supercritical cases.

In Section 2 we give some preliminary definitions and facts about Brownian motion and Brownian bridges. In Section 3 we state our main results for the conditioned process and in Section 4 we give the proofs.

2. BROWNIAN BRIDGES AND EXCURSIONS

We recall in this section some basic facts about Brownian bridges and excursions that will be used in the statements of our results.

The transition density of the process $X_t = B_t - \alpha t$ is given by

$$p^{(\alpha)}(x, y, t) = e^{-\alpha(y-x)-\alpha^2 t/2} p(x, y, t)$$

where

$$p(x, y, t) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

is the transition density of a Brownian motion.

We denote by

$$T_a^X = \inf\{t>0\colon X_t = a\}$$

the hitting time of state a. The process will start with initial condition $x>0$ and we say that the process is absorbed at time T_0^X and that the process survived till time t if $T_0^X>t$.

The semigroup of the process killed at 0 is $P^t f(x) = \mathbb{E}_x(f(X_t),\ T_0^X>t)$ and we denote by $p_-^{(\alpha)}(x, y, t)$ its associated density. We have

$$p_-^{(\alpha)}(x, y, t) = p^{(\alpha)}(x, y, t) - p^{(\alpha)}(x, -y, t) \qquad \text{for} \quad x, y>0$$

and we denote $p_-(x, y, t) \doteq p^{(0)}(x, y, t)$, so

$$p_-(x, y, t) = p(x, y, t) - p(x, -y, t)$$
$$= \frac{2}{\sqrt{2\pi t}} e^{-(x^2+y^2)/2t} \sinh\left(\frac{xy}{t}\right) \qquad \text{for} \quad x, y > 0$$

which is the transition density of a Brownian motion which does not attain 0.

When one fixes the endpoints of a path at times 0 and t at x and y, respectively, the conditional distribution

$$\mathbb{P}_x\{X \in A \mid X_t = y\} \qquad \text{for} \quad A \in \mathscr{F}_t$$

does not depend on the drift α. In fact the joint density for passing from z_1 to z_2 at times $0 \leqslant s_1 < s_2 \leqslant t$ is given by

$$q^{(x,y,t)}(z_1, z_2; s_1, s_2) = \frac{p(x, z_1, s_1)\, p(z_1, z_2, s_2 - s_1)\, p(z_2, y, t - s_2)}{p(x, y, t)}$$

When we also impose that the process does not attain 0 we have a similar result, i.e., the distribution

$$\mathbb{P}_x\{X \in A \mid T_0^X > t, X_t = y\} \qquad \text{for} \quad A \in \mathscr{F}_t$$

does not depend on the drift α and its joint density is given by

$$q_-^{(x,y,t)}(z_1, z_2; s_1, s_2) = \frac{p_-(x, z_1, s_1)\, p_-(z_1, z_2, s_2 - s_1)\, p_-(z_2, y, t - s_2)}{p_-(x, y, t)}$$

Hence when we evaluate this kind of conditional distribution we can assume that X is a Brownian motion.

The functions $q^{(x,y,t)}$ and $q_-^{(x,y,t)}$ are joint transition densities of Brownian bridges passing from x to y in $[0, t]$, the last one being such that the process dos not attain 0. Processes with such densities are denoted respectively $B^{x,y,t}$ and $B_+^{x,y,t}$; the last one is called a positive Brownian bridge between x and y in $[0, t]$. The associated distributions are denoted respectively by $BB^{x,y,t}$ and $BB_+^{x,y,t}$. When $t = 1$ we omit the dependence in t, so we write $BB_+^{x,y}$ and so on. We shall prove that it is possible to take limits, in distribution, in x and y for positive Brownian bridges $BB_+^{x,y}$.

Lemma 1. Let B be a Brownian motion. Let $x_\varepsilon, y_\varepsilon$ be strictly positive numbers for $\varepsilon > 0$ and $x_\varepsilon \to x$, $y_\varepsilon \to y$ as $\varepsilon \downarrow 0$. Then the $\mathbb{P}_{x_\varepsilon}$ conditional distribution of B given $\{T_0^B > 1, B_1 = y_\varepsilon\}$ converges in distribution as $\varepsilon \downarrow 0$.

This distribution is also called a positive Brownian bridge between x and y in $[0, 1]$, we denote it $BB_+^{x,y}$, and a process with this distribution is denoted $B_+^{x,y}$. Notice that $BB_+^{0,0}$ is a Brownian excursion in $[0, 1]$. In ref. 2 the above result was shown for the special case $x_\varepsilon = y_\varepsilon = \varepsilon$.

As usual, convergence in distribution means weak convergence on the space of continuous functions over a suitable time interval.

3. THE CONDITIONED PROCESS AND MAIN RESULTS

For a probability measure ν concentrated on $(0, \infty)$ we denote by $\mathbb{P}_\nu$ the distribution law of the process with initial distribution $\mathbb{P}_\nu\{X_0 \in C\} = \nu(C)$, i.e.,

$$\mathbb{P}_\nu\{X \in A\} = \int \mathbb{P}_x\{X \in A\}\, d\nu(x) \qquad \text{for} \quad A \in \mathscr{F}$$

The probability measure μ defined in $(0, \infty)$ by $d\mu(y) = \alpha^2 y e^{-\alpha y}\, dy$ is the limit conditional measure of (X_t) in the sense that it satisfies for any initial condition $x > 0$

$$\mu(C) = \lim_{t \to \infty} \mathbb{P}_x\{X_t \in C \mid T_0^X > t\} \qquad \text{for every Borel subset} \quad C \subset (0, \infty)$$

This measure is conditionally invariant,that is, it satisfies the following property:

$$\mu(C) = \mathbb{P}_\mu\{X_t \in C \mid T_0^X > t\} \qquad \forall t > 0 \quad \text{and for any Borel set} \quad C \subset (0, \infty)$$

The family of absolutely continuous probability measures which are conditionally invariant is a one-parameter family of left eigenvectors of the semigroup P^t associated to the diffusion (X_t); then they satisfy

$$\mu^{(\gamma)} P^t = e^{\gamma t} \mu^{(\gamma)} \qquad \text{for} \quad t > 0 \quad \text{and} \quad \gamma \in [-\alpha^2/2, 0)$$

The probability measure associated to the minimal value $\gamma_0 = -\alpha^2/2$ turns out to be the limit conditional measure, so $\mu = \mu^{(\gamma_0)}$. The other ones are given by

$$\dot{\mu}^{(\gamma)}(x) = e^{-\alpha x} \sinh(\sqrt{\alpha^2 + 2\gamma}\, x), \qquad \text{for} \quad \gamma \in (\gamma_0, 0)$$

For $\gamma \geqslant 0$, the measure $\mu^{(\gamma)}$ defined as above is also a left eigenvector and it has infinite mass. Notice that when starting from the conditionally invariant distributions, the absorption times are exponentially distributed

$$\mathbb{P}_{\mu^{(\gamma)}}\{T_0^X > t\} = e^{\gamma t}$$

and consequently ordered with respect to the standard stochastic order. The extreme distribution $\mu = \mu^{(\gamma_0)}$ is the one that has minimal absorption time. For this reason $\mu = \mu^{(\gamma_0)}$ is called the minimal quasistationary distribution.

Observe that associated to each $\gamma \geqslant \gamma_0$ there is a positive right eigenvector of the semigroup given by

$$\varphi^{(\gamma)}(x) = e^{\alpha x} \sinh(\sqrt{\alpha^2 + 2\gamma}\, x) \qquad \text{for} \quad \gamma > \gamma_0 \quad \text{and} \quad \varphi^{(\gamma_0)}(x) = xe^{\alpha x}$$

For these results see ref. 6, where many of them were shown. For a general discussion concerning limiting conditional measures and conditional invariant measures see refs. 1, 3, 6, and 9. The limit conditional distributions were firstly study by Yaglom[10] for the subcritical branching processes.

Assume that we start from some initial probability distribution ν on $(0, \infty)$. Then look for the conditional distribution of X^0 given that X survived up to t and X_t belongs to some bounded subset of $(0, \infty)$. We show that there is a phase transition concerning the region where the so conditioned distribution of X_0 is concentrated. This transition depends on the tail of ν and it takes place when the tail is of the order $e^{-\alpha x}$, where α is the decay parameter of the minimal quasistationary distribution. In fact the transition depends on some integrability condition on ν with respect to the asymptotic ratio

$$\lim_{t \to \infty} \frac{\mathbb{P}_x\{T_0^X > t\}}{\mathbb{P}_1\{T_0^X > t\}}$$

This asymptotic quantity turns to be the positive right eigenfunction associated to the minimal value γ_0. This explains why the critical rate is given by α. In the proof of our result we need some domination condition of the ratio and this is given by the following elementary result, which follows from classical estimations.[6]

Lemma 2. We have

$$\frac{\mathbb{P}_x\{T_0^X > t\}}{\mathbb{P}_y\{T_0^X > t\}} \leqslant \frac{x}{y} e^{\alpha(x-y)} e^{y^2/2t}$$

$$\leqslant K(\varepsilon) \frac{x}{y} e^{\alpha(x-y)} \qquad \text{for} \quad t \geqslant \varepsilon \quad \text{for some} \quad \varepsilon > 0$$

$$\lim_{t \to \infty} \frac{\mathbb{P}_x\{T_0^X > t\}}{\mathbb{P}_y\{T_0^X > t\}} = \frac{x}{y} e^{\alpha(x-y)}$$

In our theorems we are going to assume $y > 0$ is fixed. On the other hand, ν will be a probability measure on $(0, \infty)$. Our results deal with distributions ν satisfying one of the following three disjoint conditions, which we call (C1)–(C3).

(C1) $\int x e^{\alpha x} \nu(dx) < \infty$.

For (C2) and (C3) we assume $d\nu \ll dx$:

(C2) $d\nu/dx = k x^m e^{-\alpha x}$ for some $m \geq 0$.

(C3) $d\nu/dx = h(x)\, e^{-\theta x}$, where $\theta \in (0, \alpha)$ and the function h satisfies $\forall x > 0$:

(i) $h((\alpha-\theta)t + x\sqrt{t})/h((\alpha-\theta)t) \to_{t\to\infty} \bar{h}$ for some constant $\bar{h} \in (0, \infty)$.

(ii) $h((\alpha-\theta)t + x\sqrt{t})/h((\alpha-\theta)t) \leq g(x)$ for some function g satisfying

$$\int_0^\infty g(x)\, e^{-x^2/2 + \varepsilon x}\, dx < \infty \qquad \text{for some} \quad \varepsilon > 0$$

We observe that the class of functions h which satisfy condition (C3)(i), (ii) includes all the polynomials, and also all the finite combinations of the form $\sum_i (x + a_i)^{m_i}$, where $a_i > 0$ if $m_i < 0$. It cannot contain exponentials, but it contains functions between polynomials and exponentials, for instance, $h(x) = x^{\log x}$ for large x.

Theorem 1. For ν satisfying one of the above conditions we have:

(C1)
$$\lim_{t\to\infty} \mathbb{P}_\nu\{X_0 \leq x \mid X_t = y, T_0^X > t\}$$
$$= \left(\int_0^\infty u e^{\alpha u}\, \nu(du)\right)^{-1} \int_0^x u e^{\alpha u}\, \nu(du) \qquad \text{for} \quad x > 0$$

If also $d\nu \ll dx$, then

$$\lim_{t\to\infty} \frac{d}{dx} \mathbb{P}_\nu\{X_0 \leq x \mid X_t = y, T_0^X > t\} = \frac{x e^{\alpha x}\, d\nu/dx}{\int x' e^{\alpha x'}\, \nu(dx')}$$

(C2)
$$\lim_{t\to\infty} \frac{d}{dx} \mathbb{P}_\nu\{X_0 \leq x\sqrt{t} \mid X_t = y, T_0^X > t\}$$
$$= c(m)\, x^{m+1} e^{-x^2/2} \qquad \text{for} \quad x > 0$$

where $c(m) = (\int_0^\infty u^{m+1} e^{-u^2/2}\, du)^{-1}$.

(C3) $$\lim_{t\to\infty} \frac{d}{dx} \mathbb{P}_\nu\{X_0 \leqslant (\alpha-\theta)t + x\sqrt{t} \mid X_t = y, T_0^X > t\})$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{for} \quad x \in \mathbb{R}$$

The case (C2) is called critical and cases (C1) and (C3) are called respectively subcritical and supercritical.

Now we study the conditional distribution $\mathbb{P}_x\{X \in A \mid X_t = y\}$ for $A \in \mathscr{F}_t$, properly localized and in the scale $\sqrt{t}$. For treating the three cases (C1)–(C3) with a general approach we are going to put $\theta = \alpha$ in case (C1) or (C2) and $\theta < \alpha$ in case (C3). We shall consider the process

$$Z_u = \frac{1}{\sqrt{t}}(X_{tu} - (\alpha-\theta)\, t(1-u)) \qquad \text{for} \quad u \geqslant 0$$

which is a Brownian motion with drift. We will study the limit distribution of this process for $u \in [0, 1]$ conditioned on the event $\{X_t = y, T_0^X > t\}$. We have $\{X_t = y\} \Leftrightarrow \{Z_1 = y/\sqrt{t}\}$. By defining

$$S^{Z,t} = \inf\{u > 0 \colon Z_u \leqslant -(\alpha-\theta)\sqrt{t}(1-u)\}$$

we get $\{T_0^X > t\} \Leftrightarrow \{S^{Z,t} > 1\}$. On the other hand, $X_0 = (Z_0\sqrt{t} + (\alpha-\theta)t)$, so that $X_0 \sim \nu$ is equivalent to $Z_0 \sim \nu'$, where

$$\nu'(z, z+dz) = \nu((z\sqrt{t} + (\alpha-\theta)t), ((z+dz)\sqrt{t} + (\alpha-\theta)t))$$

Then, studying the $\mathbb{P}_\nu$ limit distribution of $(X_{t\bullet} - (\alpha-\theta)\, t(1-\bullet))/\sqrt{t}$ with $\bullet \in [0, 1]$ conditioned on the event $\{X_t = y, T_0^X > t\}$ is equivalent to studying

$$\lim_{t\to\infty} \mathbb{P}_{\nu'}\{Z_\bullet \in A \mid S^{Z,t} > 1, Z_1 = y/\sqrt{t}\} \tag{1}$$

for A in the σ-field generated by the coordinates belonging to $[0, 1]$.

In the next result we use an extended notion of Brownian bridges. Even if the initial point is not fixed, we continue to call it a Brownian bridge; thus the distribution of $\mathbb{P}_\tau\{B \in A \mid B_1 = y\}$ is denoted $BB^{\tau,y}$, and if τ is concentrated on $(0, \infty)$, the distribution $\mathbb{P}_\tau\{B \in A \mid T_0^B > 1, B_1 = y\}$ is denoted $BB_+^{\tau,y}$.

Theorem 2. The limit conditional distribution given in (1) is the following one in the three different cases:

(C1) A Brownian excursion on $[0, 1]$.

(C2) A Brownian bridge conditioned to be positive in $[0, 1)$, $BB_+^{\tau,0}$, where τ has density $c(m)x^{m+1}e^{-x^2/2}\cdot 1_{\{x>0\}}$.

(C3) A Brownian bridge $BB^{\tau,0}$, where τ has Gaussian density $e^{-x^2/2}/\sqrt{2\pi}$.

We remark that result (C3) is equivalent to saying that when we reverse the time on the limit distribution, i.e., we make $u'=1-u$ for $u\in[0,1]$, it is a Brownian motion starting from 0. On the other hand, (C3) implies that in the supercritical case $(X_{st}/t)_{s\in[0,1]}$ conditioned on $\{T_0^X>t\}$ converges in distribution to the line $y(s)=(\alpha-\theta)(1-s)$.

4. PROOF OF RESULTS

Proof of Lemma 1. We prove this result only in the case $x=y=0$. We follow the method developed in ref. 2. The proof is divided into two parts. First we prove that the finite-dimensional distributions converge. Second we prove that the set of measures $\mathbb{P}_{x_\varepsilon}\{\bullet \mid T_0^Z>1, Z_1=y_\varepsilon\}$ is tight.

The convergence of the finite-dimensional distribution follows from the Markov property and the limit of the joint density is given by

$$q_-(z_1, z_2, s_1, s_2) \doteq \lim_{\substack{x'\to 0\\ y'\to 0}} q_-^{(x',y',1)}(z_1, z_2, s_1\ s_2)$$
$$= \frac{2z_1z_2e^{-(z_1^2/2s_1+z_2^2/2(1-s_2))}}{\sqrt{2\pi}(s_1(1-s_2))^{3/2}}\, p_-(z_1, z_2, s_2-s_1)$$

which corresponds to the transition density of a Brownian excursion. This last limit is easily computed by using the l'Hôpital rule.

We note that the marginal conditional density is given by

$$q_-(z,s) \doteq \lim_{\substack{x'\to 0\\ y'\to 0}} \frac{p_-(x',z,s)\,p_-(z,y',t-s)}{p_-(x',y',t)} = \frac{2z^2e^{-z^2/2s(1-s)}}{\sqrt{2\pi}(s(1-s))^{3/2}}$$

For tightness we shall prove that for every $1/2>\delta>0$ the induced measures on $\mathscr{C}[\delta, 1-\delta]$ by $\mathbb{P}_{x_\varepsilon}\{\bullet \mid T_0^Z>1, Z_1=y_\varepsilon\}$ is a tight family, and that for every $\eta>0$

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\mathbb{P}_{x_\varepsilon}\{\sup_{0\leqslant u\leqslant\delta}|Z_u|<\eta \mid T_0^Z>1, Z_1=y_\varepsilon\}=1 \tag{2}$$

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\mathbb{P}_{x_\varepsilon}\{\sup_{1-\delta\leqslant u\leqslant 1}|Z_u|<\eta \mid T_0^Z>1, Z_1=y_\varepsilon\}=1 \tag{3}$$

The corresponding tightness on $\mathscr{C}[0,1]$ follows from Theorem (3.1) in ref. 2.

Let us fix $1/2>\delta>0$. For every $\varepsilon>0$, consider a compact set K of $\mathscr{C}[\delta, 1-\delta]$ such that $BB_+^{0,0}(\pi^{-1}K)\geqslant 1-\varepsilon$, where $\pi: \mathscr{C}[0,1]\to\mathscr{C}[\delta,1-\delta]$ is the natural projection and $BB_+^{0,0}$ is the distribution of a Brownian excursion in $[0,1]$. The Markov property shows that

$$\mathbb{P}_{x_\varepsilon}\{Z\in\pi^{-1}K \mid T_0^Z>1, Z_1=y_\varepsilon\}$$
$$=\int_0^\infty\int_0^\infty \frac{p_-(x_\varepsilon,u,\delta)\,p_-(v,y_\varepsilon,\delta)}{p_-(x_\varepsilon,y_\varepsilon,1)}\,BB_+^{u,v,1-2\delta}(K)\,du\,dv$$

We remind the reader that $BB_+^{u,v,1-2\delta}$ is the distribution of a positive Brownian bridge on $[0,1-2\delta]$ with u as initial position and v as final position.

By l'Hôpital rule and Scheffe's lemma[7] we deduce

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}_{x_\varepsilon}\{Z\in\pi^{-1}K \mid T_0^Z>1, Z_1=y_\varepsilon\}=BB_+^{0,0}(\pi^{-1}K)$$

from which the tightness in $\mathscr{C}[\delta,1-\delta]$ follows.

We shall now prove (2). First notice that

$$J(\varepsilon,\delta)=\mathbb{P}_{x_\varepsilon}\{\sup_{0\leqslant s\leqslant\delta}|Z_s|<\eta \mid T_0^Z>1, Z_1=y_\varepsilon\}$$
$$=\int_0^\eta \mathbb{P}_{x_\varepsilon}\{\sup_{0\leqslant s\leqslant\delta}|Z_s|<\eta, T_0^Z>\delta, Z_\delta\in dz\}\,\frac{p_-(z,y_\varepsilon,1-\delta)}{p_-(x_\varepsilon,y_\varepsilon,1)}$$
$$=\int_0^\eta \mathbb{P}_{x_\varepsilon}\{T_\eta^Z>\delta \mid T_0^Z>\delta, Z_\delta=z\}\,\frac{p_-(z,y_\varepsilon,1-\delta)\,p_-(x_\varepsilon,z,\delta)}{p_-(x_\varepsilon,y_\varepsilon,1)}\,dz$$

Using Proposition 2.8.10 from ref. 5, we obtain

$$\mathbb{P}_{x_\varepsilon}\{T_\eta^Z>\delta \mid T_0^Z>\delta, Z_\delta=z\}=\sum_{n\in\mathbb{Z}}\frac{p_-(x_\varepsilon,z+2n\eta,\delta)}{p_-(x_\varepsilon,z,\delta)}$$

where $p_-(x,y,t)=-p_-(x,|y|,t)$ when $x>0$ and $y<0$.

Therefore

$$J(\varepsilon,\delta)=\int_0^\eta\left\{1+\sum_{n\neq 0}\frac{p_-(x_\varepsilon,z+2n\eta,\delta)}{p_-(x_\varepsilon,z,\delta)}\right\}q_-^{(x_\varepsilon,y_\varepsilon,1)}(z,\delta)\,dz$$

We have the following estimates:

$$q_-^{(x_\varepsilon, y_\varepsilon, 1)}(z, \delta)$$

$$= \frac{2\exp[-(x_\varepsilon^2/2)(1-\delta)/\delta - (y_\varepsilon^2/2)\,\delta/(1-\delta)]}{\sqrt{2\pi}\,[\delta(1-\delta)]^{1/2}}$$

$$\times\left(\frac{\sinh(x_\varepsilon z/\delta)\sinh(y_\varepsilon z/(1-\delta))\exp[-z^2/2\delta(1-\delta)]}{\sinh(x_\varepsilon y_\varepsilon)}\right)$$

$$= A\,\frac{\cosh(\xi)\cosh(\xi')}{\cosh(\xi'')}\,\frac{2z^2\exp[-z^2/2\delta(1-\delta)]}{\sqrt{2\pi}[\delta(1-\delta)]^{3/2}}$$

where

$$A = \exp\left[-\left(\frac{x_\varepsilon^2}{2}\frac{1-\delta}{\delta} + \frac{y_\varepsilon^2}{2}\frac{\delta}{1-\delta}\right)\right]$$

$$|\xi| \leqslant \frac{x_\varepsilon z}{\delta}, \qquad |\xi'| \leqslant \frac{y_\varepsilon z}{1-\delta}, \qquad |\xi''| \leqslant x_\varepsilon y_\varepsilon$$

Therefore

$$q_-^{(x_\varepsilon, y_\varepsilon, 1)} \xrightarrow[\varepsilon\downarrow 0]{} q_-(z, \delta)$$

Moreover

$$q_-^{(x_\varepsilon, y_\varepsilon, 1)}(z, \delta) \leqslant \frac{D}{\delta^{3/2}}\,e^{a\eta/\delta} z^2 e^{-z^2/2\delta(1-\delta)} \tag{4}$$

for some constants D and a depending only on η, for small ε.

On the other hand,

$$\frac{p_-(x_\varepsilon, z+2n\eta, \delta)}{p_-(x_\varepsilon, z, \delta)}$$

$$= e^{-2n^2\eta^2/\delta - 2zn\eta/\delta}\left(\frac{\sinh[(x_\varepsilon/\delta)(z+2n\eta)]}{\sinh(x_\varepsilon z)}\right)$$

$$= e^{-2n^2\eta^2/\delta - 2zn\eta/\delta}\left(\frac{\cosh(\xi)}{\cosh(\xi')}\cdot\frac{z+2n\eta}{z}\right)$$

for some $|\xi| \leqslant x_\varepsilon|z+2n\eta|/\delta$, $|\xi'| \leqslant x_\varepsilon z$.

Hence,

$$\lim_{\varepsilon \downarrow 0} \frac{p_-(x_\varepsilon, z+2n\eta, \delta)}{p_-(x_\varepsilon, z, \delta)} = e^{-2n^2\eta^2/\delta - 2zn\eta/\delta} \left(\frac{z+2n\eta}{z}\right) \tag{5}$$

Also, we have for $0 < z \leqslant \eta$

$$\left|\frac{p_-(x_\varepsilon, z+2n\eta, \delta)}{p_-(x_\varepsilon, z, \delta)}\right| \leqslant \frac{C\,|n|}{z} e^{-2n^2\eta^2/\delta + \gamma\eta\,|n|/\delta} \tag{6}$$

where C and γ depend only on η, for small ε.

From (5) and (6) we deduce that

$$0 \leqslant \lim_{\varepsilon \downarrow 0} \mathbb{P}_{x_\varepsilon}\{T_\eta^Z > \delta \mid T_0^Z > \delta, Z_\delta = z\} = 1 + \sum_{n \neq 0} e^{-2n^2\eta^2/\delta - 2zn\eta/\delta} \left(\frac{z+2n\eta}{z}\right)$$

Using (4)–(6) and the dominated converge theorem, we get

$$\begin{aligned}\lim_{\varepsilon \downarrow 0} J(\varepsilon, \delta) &= \int_0^\eta \left(1 + \sum_{n \neq 0} e^{-2n^2\eta^2/\delta - 2zn\eta/\delta} \left(\frac{z+2n\eta}{z}\right)\right) \frac{2z^2 e^{-z^2/2\delta(1-\delta)}}{\sqrt{2\pi}\,[\delta(1-\delta)]^{3/2}}\, dz \\ &\geqslant \int_0^{\eta/2} q_-(z, \delta)\, dz - \int_0^{\eta/2} \sum_{n \neq 0} e^{-2n^2\eta^2/\delta + |n|\,\eta^2/\delta} \\ &\quad \times (1 + 2\,|n|)\, \eta \frac{2z e^{-z^2/2\delta(1-\delta)}}{\sqrt{2\pi}\,[\delta(1-\delta)]^{3/2}}\, dz\end{aligned}$$

Since

$$\int_0^{\eta/2} q_-(z, \delta)\, dz \xrightarrow[\delta \to 0]{} 1$$

and the last term is bounded by

$$\frac{3\eta}{[\delta(1-\delta)]^{1/2}} \sum_{n \neq 0} |n|\, e^{-2n^2\eta^2/\delta + |n|\,\eta^2/\delta} \xrightarrow[\delta \to 0]{} 0$$

the relation (2) follows. The proof of (3) is completely analogous. ∎

Proof of Theorem 1.

(C1) $\mathbb{P}_\nu\{X_0 \leqslant x \mid X_t = y, T_0^X > t\} = \dfrac{A(t, x)}{A(t, \infty)}$

where

$$A(t,x)=\int_0^x \mathbb{P}_u\{X_t\in dy \mid T_0^X>t\}\cdot\frac{\mathbb{P}_u\{T_0^X>t\}}{\mathbb{P}_1\{T_0^X>t\}}\,dv(u)$$

By using Lemma 2 and the fact that μ is the limit conditional measure, i.e.,

$$\mathbb{P}_u\{X_t\in dy \mid T_0^X>t\}\xrightarrow[t\to\infty]{}\mu(dy)$$

we can pass to the limit $t\to\infty$ in $A(t,x)$ to obtain

$$\lim_{t\to\infty} A(t,x)=\mu(dy)\int_0^x ue^{\alpha u}\,v(du)$$

Since this also holds for $x=\infty$, the result is shown.

Let us show the second statement in (C1). We denote $f=dv/dx$ [this notation will be also used in (C2) and (C3)]. We have

$$\begin{aligned}&\frac{d}{dx}\mathbb{P}_v\{X_0\leqslant x \mid X_t=y,\, T_0^X>t\}\\&=\frac{f(x)\,p_-^{(\alpha)}(x,y,t)}{\int f(x')\,p_-^{(\alpha)}(x',y,t)\,dx'}\\&=\frac{f(x)\,e^{\alpha x}e^{-(1/2t)x^2}\sinh(xy/t)}{\int f(x')\,e^{\alpha x'}e^{-(1/2t)x'^2}\sinh(x'y/t)\,dx'}\end{aligned}$$

When $t\to\infty$, both terms, numerator and denominator, converge to 0. Take $u=1/t$; in order to be able to apply the l'Hôpital rule, observe that

$$\frac{d}{du}\left(e^{-(u/2)x'^2}\sinh(x'yu)\right)\leqslant Kx',\qquad \forall x'>0$$

with $K=K(\varepsilon)$ and uniform in $u\in(0,\varepsilon)$, for some small $\varepsilon>0$. Hence, from our hypothese we can apply the l'Hôpital rule and obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{d}{dx}\mathbb{P}_v\{X_0\leqslant x \mid X_t=y,\, T_0^X>t\}\\&=\lim_{u\to 0}\frac{f(x)\,e^{\alpha x}\left(-\frac{1}{2}x^2\sinh(xyu)+xy\cosh(xyu)\right)e^{-ux^2/2}}{\int f(x')\,e^{\alpha x'}\left(-\frac{1}{2}x'^2\sinh(x'yu)+x'y\cosh(x'yu)\right)e^{-ux'^2/2}\,dx'}\\&=\frac{f(x)\,e^{\alpha x}x}{\int f(x')\,e^{\alpha x'}x'\,dx'}\end{aligned}$$

Now, let us show (C2) and (C3).

$$\text{(C2)}\quad \frac{d}{dx}\mathbb{P}_\nu\{X_0 \leqslant x\sqrt{t} \mid T_0^X > t, X_t = y\}$$

$$= \frac{f(x\sqrt{t})\, p_-^{(\alpha)}(\sqrt{t}\,x, y, t)\, dy}{\int \mathbb{P}_{x'\sqrt{t}}\{T_0^X > t, X_t \in dy\}\, f(x'\sqrt{t})\, dx'}$$

Easy computations show that this quantity is equal to

$$\frac{x^m p_-(x, y/\sqrt{t}, 1)}{\int x'^m p_-(x', y/\sqrt{t}, 1)\, dx'} = \frac{x^m e^{-x^2/2} \sinh(xy/\sqrt{t})}{\int x'^m e^{-x'^2/2} \sinh(x'y/\sqrt{t})\, dx'}$$

We can apply the l'Hôpital rule to get that

$$\lim_{t\to\infty} \frac{d}{dx}\mathbb{P}_\nu\{X_0 \leqslant x\sqrt{t} \mid T_0^X > t, X_t = y\} = \frac{x^{m+1} e^{-x^2/2}}{\int_0^\infty x'^{m+1} e^{-x'^2/2}\, dx'}$$

Now we consider (C3).

$$\text{(C3)}\quad \frac{d}{dx}\mathbb{P}_\nu\{X_0 \leqslant (\alpha-\theta)t + x\sqrt{t} \mid T_0^X > t, X_t = y\}$$

$$= \frac{f((\alpha-\theta)t + \sqrt{t}\,x)\, p_-^{(\alpha)}((\alpha-\theta)t + \sqrt{t}\,x, y, t)}{\int f((\alpha-\theta)t + \sqrt{t}\,x')\, p_-^{(\alpha)}((\alpha-\theta)t + \sqrt{t}\,x', y, t)\, dx'}$$

$$= \frac{h((\alpha-\theta)t + \sqrt{t}\,x)\, e^{-(\alpha-\theta)x\sqrt{t}}\, p_-((\alpha-\theta)t + x\sqrt{t}, y, t)}{\int h((\alpha-\theta)t + \sqrt{x'})\, e^{-(\alpha-\theta)x'\sqrt{t}}\, p_-((\alpha-\theta)t + x'\sqrt{t}, y, t)\, dx'}$$

By developing the p_- term we find that this expression is equal to

$$\frac{h((\alpha-\theta)t + x\sqrt{t})\, e^{-x^2/2} \sinh\{[(\alpha-\theta) + x/\sqrt{t}]\, y\}}{\int h((\alpha-\theta)t + x'\sqrt{t})\, e^{-x'^2/2} \sinh\{[(\alpha-\theta) + x'/\sqrt{t}]\, y'\}\, dx'}$$

Dividing the numerator and denominator by $h((\alpha-\theta)t)$, making $t \to \infty$, and using our hypotheses on the function h, we find

$$\frac{d}{dx}\mathbb{P}_\nu\{X_0 \leqslant (\alpha-\theta)t + x\sqrt{t} \mid T_0^X > t, X_t = y\} = \frac{e^{-x^2/2}}{\int e^{-x'^2/2}\, dx'} \quad \blacksquare$$

Remark. From Scheffé's lemma, under condition (C2), we have $\mathbb{P}_\nu\{X_0 \leqslant x\sqrt{t} \mid T_0^X > t, X_t = y\}$ converges in distribution and its limit has a density given by Theorem 1. A similar comment holds under (C3).

Proof of Theorem 2. (C1) In this case $Z_u = X_{ut}/\sqrt{t}$, and we obtain

$$\mathbb{P}_\nu\left\{\frac{X_{\bullet t}}{\sqrt{t}} \in A \mid X_t = y,\, T_0^X > t\right\}$$

$$= \left[\int \mathbb{P}_{x/\sqrt{t}}\{Z \in A \mid Z_1 = y/\sqrt{t},\, T_0^Z > 1\}\right.$$

$$\left. \cdot\, \mathbb{P}_x\{X_t \in dy \mid T_0^X > t\}\, \frac{\mathbb{P}_x\{T_0^X > t\}}{\mathbb{P}_1\{T_0^X > t\}}\, \nu(dx)\right]$$

$$\times \left[\int \mathbb{P}_x\{X_t \in dy \mid T_0^X > t\} \cdot \frac{\mathbb{P}_x\{T_0^X > t\}}{\mathbb{P}_1\{T_0^X > t\}}\, \nu(dx)\right]^{-1}$$

Since

$$\lim_{t \to \infty} \mathbb{P}_{x/\sqrt{t}}\{Z \in A \mid Z_1 = y/\sqrt{t},\, T_0^Z > 1\} = BB_+(A)$$

$$\lim_{t \to \infty} \mathbb{P}_x\{X_t \in dy \mid T_0^X > t\} = \alpha^2 y e^{-\alpha y}\, dy$$

$$\frac{\mathbb{P}_x\{T_0^X > t\}}{\mathbb{P}_1\{T_0^X > t\}} \leqslant K x e^{\alpha x} \qquad \text{for all } t \text{ big enough, which is } \nu\text{-integrable}$$

and

$$\lim_{t \to \infty} \frac{\mathbb{P}_x\{T_0^X > t\}}{\mathbb{P}_1\{T_0^X > t\}} = x e^{\alpha x}$$

the result follows from the dominated convergence theorem.

Let us show the result for the critical and supercritical cases (C2) and (C3).

Let $F\colon \mathscr{C}[0, 1] \to \mathbb{R}$ be a bounded continuous function. We must study the limit of the quantity

$$\mathbb{E}_\nu(F(Z) \mid T_0^X > t,\, X_t = y)$$

Denote

$$G_t(z) = \mathbb{E}_z(F(Z) \mid T_0^X > t,\, Z_1 = y/\sqrt{t})$$

$$h_t(z) = \frac{f(z)\, \mathbb{P}_z\{T_0^X > t,\, Z_1 \in dy/\sqrt{t}\}}{\int f(z')\, \mathbb{P}_{z'}\{T_0^X > t,\, Z_1 \in dy/\sqrt{t}\}\, dz'}$$

where $f = d\nu/dx$.

From Theorem 1 we have that $h_t(z) \to_{t\to\infty} h(z)$ pointwise, where

$$h(z) = \begin{cases} x^{m+1} c(m)\, z^{m+1} e^{-z^2/2} & \text{for } z>0 \text{ in the critical case} \\ e^{-z^2/2} (\int e^{-z'^2/2}\, dz')^{-1} & \text{in the supercritical case} \end{cases}$$

Observe that $|G_t(z)| \leqslant \|F\|_\infty$, so if we are able to show that $G_t(z)$ converge pointwise to a $G(z)$, then

$$\int h_t(z)\, G_t(z)\, dz \xrightarrow[t\to\infty]{} \int h(z)\, G(z)\, dz$$

will follow from an application of Scheffé's lemma. The identification of the limit will be made by inspection on G.

For studying $G_t(z)$ we observe that Z can be assumed to be a Brownian motion (starting from z) because it is a conditional distribution with both extremities fixed.

In the critical case we have $Z_s = X_{st}/\sqrt{t}$ and so $T_0^X > t \Leftrightarrow T_0^Z > 1$. From Lemma 1 we have

$$\forall z>0: \quad \mathbb{E}_z(F(Z) \mid T_0^Z > 1, Z_1 = y/\sqrt{t}) \xrightarrow[t\to\infty]{} \mathbb{E}_z(F(B_+^{z,0}))$$

where $B_+^{z,0}$ is a Brownian bridge from z to 0 which does not attain 0 before time 1.

Let us study the supercritical case and instead of $G_t(Z)$ we will study

$$\mathbb{E}_z(Z\in A \mid T_0^X > t, Z_1 = y/\sqrt{t}) \qquad \text{for} \quad A \subseteq \mathscr{F}_r \quad \text{with} \quad r<1$$

Now

$$\begin{aligned} &\mathbb{E}_z(Z\in A, S^{Z,t} > 1, Z_1 \in dy/\sqrt{t}) \\ &\quad = \mathbb{E}_z(Z\in A, S^{Z,t} > r, \mathbb{E}_{Z_r}(\bar{S}^{Z,t} > 1-r, Z_{1-r} \in dy/\sqrt{t})) \end{aligned}$$

where $\bar{S}^{Z,t} = \inf\{s>0: Z_s \geqslant -(\alpha-\theta)\sqrt{t}\,(1-(r+s))\}$.

Hence the above expression is

$$\int \mathbb{E}^z(Z\in A, S^{Z,t} > r \mid Z_r = \lambda)$$

$$\times\, \mathbb{E}_\lambda(\bar{S}^{Z,t} > 1-r, Z_{1-r} \in dy/\sqrt{t})\, \frac{e^{-(\lambda-z)^2/2r}}{\sqrt{2\pi r}}\, d\lambda$$

Let us study the first term in the integral. For this purpose we shall show that

$$\mathbb{E}_z(S^{Z,t} > r \mid Z_r = \lambda) \xrightarrow[t \to \infty]{} 1$$

Make the change of variables $U_s = Z_s + (\alpha - \theta)\sqrt{t}(1-s)$. We have $U_0 = z + (\alpha - \theta)\sqrt{t}$, $U_r = \lambda + (\alpha - \theta)\sqrt{t}(1-r)$, and $S^{Z,t} > r$ is equivalent to $T_0^U > r$. Then

$$\begin{aligned}
&\mathbb{E}_z(S^{Z,t} > r \mid Z_s = \lambda)\\
&\quad = \mathbb{E}(T_0^U > r \mid U_0 = z + (\alpha - \theta)\sqrt{t},\ U_r = \lambda + (\alpha - \theta)\sqrt{t}(1-r))
\end{aligned}$$

Since it is a conditional distribution with fixed extrema, we can assume U is a Brownian motion, and we find that the last expression is

$$\begin{aligned}
&\frac{p_-(z + (\alpha-\theta)\sqrt{t}, \lambda + (\alpha-\theta)\sqrt{t}(1-r), r)}{p(z + (\alpha-\theta)\sqrt{t}, \lambda + (\alpha-\theta)\sqrt{t}(1-r), r)}\\
&\quad = 1 - \exp\left[-\frac{2}{r}(z + (\alpha-\theta)\sqrt{t})(\lambda + (\alpha-\theta)\sqrt{t}(1-r))\right] \xrightarrow[t \to \infty]{} 1
\end{aligned}$$

Hence

$$\mathbb{E}_z(Z \in A, S^{Z,t} > r \mid Z_r = \lambda) \xrightarrow[t \to \infty]{} \mathbb{E}_z(Z \in A \mid Z_r = \lambda)$$

On the other hand, make the change of variable $Y_u = Z_u + (\alpha - \theta)(1 - (r+u))\sqrt{t}$ to obtain

$$\begin{aligned}
&\mathbb{E}_\lambda(\bar{S}^{Z,t} > 1 - r, Z_{1-r} \in dy/\sqrt{t})\\
&\quad = \mathbb{E}_{\lambda + (\alpha-\theta)(1-r)\sqrt{t}}(T_0^Y > 1 - r,\ Y_{1-r} \in dy/\sqrt{t})\\
&\quad = p_-^{(\alpha-\theta)\sqrt{t}}(\lambda + (\alpha-\theta)(1-r)\sqrt{t}, y/\sqrt{t}, 1-r)\, dy
\end{aligned}$$

which yields

$$\begin{aligned}
&\mathbb{E}_\lambda(\bar{S}^{Z,t} > 1 - r, Z_{1-r} \in dy/\sqrt{t})\\
&\quad = \frac{1}{\sqrt{2\pi(1-r)}} e^{-y(\alpha-\theta)} e^{-y^2/2t(1-r)} e^{-\lambda^2/2(1-r)}\\
&\qquad \times \sinh\left(\frac{\lambda y}{\sqrt{t}(1-r)} + (\alpha-\theta)\, y\right) dy
\end{aligned}$$

By taking $\lambda = z$, $r = 0$ we also get

$$\mathbb{E}_z(S^{Z,t} > 1, Z_1 \in dy/\sqrt{t})$$
$$= \frac{1}{\sqrt{2\pi}} e^{-y(\alpha-\theta)} e^{-y^2/2t} e^{-z^2/2} \sinh\left(\frac{zy}{\sqrt{t}} + (\alpha - \theta)\, y\right) dy$$

Then,

$$\frac{\mathbb{E}_\lambda(\bar{S}^{Z,t} > 1 - r, Z_{1-r} \in dy/\sqrt{t})}{\mathbb{E}_z(S^{Z,t} > 1, Z_1 \in dy/\sqrt{t})} \xrightarrow[t \to \infty]{} \frac{e^{-\lambda^2/2(1-r) + z^2/2}}{\sqrt{1-r}}$$

and we also get a domination in λ, uniform for large t, which allows to integrate in our formula.

Then

$$\mathbb{E}_z(Z \in A \mid T_0^X > t, Z_1 = y/\sqrt{t})$$
$$= \mathbb{E}_z(Z \in A \mid S^{Z,t} > 1, Z_1 = y/\sqrt{t})$$
$$= \int \mathbb{E}_z(Z \in A, S^{Z,t} > r \mid Z_r = \lambda)$$
$$\times \frac{\mathbb{E}_\lambda(\bar{S}^{Z,t} > 1 - r, Z_{1-r} \in dy/\sqrt{t})}{\mathbb{E}_z(S^{Z,t} > 1, Z_1 \in dy/\sqrt{t})} \frac{e^{-(\lambda - z)^2/2r}}{\sqrt{2\pi r}} d\lambda$$
$$\xrightarrow[t \to \infty]{} \int \mathbb{E}_z(Z \in A \mid Z_r = \lambda) \frac{e^{-(\lambda - z)^2/2r}}{\sqrt{2\pi r}} \frac{e^{-\lambda^2/2r(1-r) + z^2/2}}{\sqrt{1-r}} d\lambda$$
$$= \int \mathbb{E}_z(Z \in A \mid Z_r = \lambda) \frac{p(z, \lambda, r)\, p(\lambda, 0, 1-r)}{p(z, 0, 1)} d\lambda$$

Therefore,

$$\forall A \in \bigcup_{s<1} \mathscr{F}_s: \quad \lim_{t \to \infty} \mathbb{P}_y\{Z \in A \mid T_0^X > t, Z_t = y\} = \mathbb{P}\{W \in A \mid W_1 = 0\}$$

where W is a Brownian motion with a standard Gaussian initial distribution.

To finish the proof it suffices to show tightness. Following Theorem 3.1 of ref. 2, it suffices to study the process around the extremity $Z_1 = y/\sqrt{t}$. We are led to prove for any $z > 0$

$$\lim_{\delta \downarrow 0} \lim_{t \to \infty} \mathbb{P}_z\{\sup_{1-\delta \leqslant s \leqslant 1} |Z_s - y/\sqrt{t}| > \varepsilon \mid S^{Z,t} > 1, Z_1 = y/\sqrt{t}\} = 0$$

We have

$$\mathbb{P}_z\{S^{Z,t} > 1 \mid Z_1 = y/\sqrt{t}\} \xrightarrow[t \to \infty]{} 1 - e^{-2(\alpha - \theta) y} > 0$$

Then it is sufficient to show

$$\lim_{\delta \downarrow 0} \lim_{t \to \infty} \mathbb{P}_z\{ \sup_{1-\delta \leqslant r \leqslant 1} |Z_s - y/\sqrt{t}| > \varepsilon \mid Z_1 = y/\sqrt{t}\} = 0$$

By making the change of variable $Y_s = Z_{1-s} - y/\sqrt{t}$ we have

$$\begin{aligned} &\mathbb{P}_z\{ \sup_{1-\delta \leqslant r \leqslant 1} |Z_s - y/\sqrt{t}| > \varepsilon \mid Z_1 = y/\sqrt{t}\} \\ &\quad = \mathbb{P}_0\{T^Y_{-\varepsilon} \wedge T^Y_\varepsilon < \delta \mid Y_1 = z - y/\sqrt{t}\} \\ &\quad \xrightarrow[t \to \infty]{} \mathbb{P}_0(T^Y_{-\varepsilon} \wedge T^Y_\varepsilon < \delta \mid Y_1 = z) \end{aligned}$$

This converges to 0 when $\delta \downarrow 0$ because the Brownian bridge has continuous trajectories. ∎

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